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Note

Uniform Convergence of Modified Hermite–Fejér Interpolation Process Omitting Derivatives

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1. INTRODUCTION

The well-known Hermite–Fejér interpolation process for a function $f(x)$ is given by

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k)\, h_k(x), \tag{1.1}$$

where

$$H_n(f, x_k) = f(x_k), \qquad H_n'(f, x_k) = 0, \qquad k = 1, 2,..., n. \tag{1.2}$$

The fundamental functions $h_k(x)$ in (1.1) are given by

$$h_k(x) = \left[1 - \frac{W_n''(x_k)}{W_n'(x_k)}(x - x_k)\right] l_k^2(x), \qquad k = 1, 2,..., n, \tag{1.3}$$

where

$$l_k(x) = \frac{W_n(x)}{(x - x_k)\, W_n'(x_k)} \tag{1.4}$$

and $\{x_k\}_{k=1}^{n}$ are the zeros of a polynomial $W_n(x)$:

$$1 \geqslant x_1 > x_2 > \cdots > x_n \geqslant -1 \qquad (n = 1, 2,...). \tag{1.5}$$

According to Fejér [1], $H_n(f, x)$, with $W_n(x) = T_n(x)$, the nth Tchebycheff polynomial of the first kind, converges uniformly to $f(x) \in C[-1, 1]$. In 1960 Turán suggested that perhaps omission of derivatives at a "few" exceptional points η_ν would not damage the convergence property of the resulting modified Hermite–Fejér polynomial $H_{\nu(n)}^*(f, x)$, now of a lower degree than

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$H_n(f, x)$. In [6] he proved the unexpected result that the convergence of $H^*_{\nu(n)}(f, x)$ is not uniform in general. Uniform convergence in $[-1, 1]$ holds iff

$$\int_{-1}^{1} \frac{xf(x)}{(1-x^2)^{1/2}}\, dx = 0 \tag{1.6}$$

when the interpolation nodes are the zeros of $T_n(x)$. But $\lim H^*_{\nu(n)}(f, x)_{x=\cos \pi/5}$ does not exist for a suitable continuous function when the exceptional point is near to $\cos \pi/5$. For detailed study one is referred to [3, 6, 7, 8]. Now one may ask the following question:

Is there any matrix of nodes for which the modified Hermite–Fejér interpolation process $H^*_{\nu(n)}(f, x)$ given by

$$H^*_{\nu(n)}(f, x) = H_n(f, x) + (x - x_\mu)\, l_\mu^2(x)\, W_n'^2(x_\mu) \sum_{k=1}^{n} f(x_k) \frac{W_n''(x_k)}{W_n'^3(x_k)}, \tag{1.7}$$

satisfying the properties

$$\begin{aligned} H^*_{\nu(n)}(f, x_k) &= f(x_k), \qquad k = 1, 2,\ldots, n, \\ H^{*\prime}_{\nu(n)}(f, x_k) &= 0, \qquad 1 \leqslant k \leqslant n, \quad k \neq \mu, \end{aligned} \tag{1.8}$$

converges uniformly to every $f(x) \in C[-1, 1]$? We shall answer this question in the affirmative by proving the

THEOREM. *The interpolation process $H^*_{\nu(n)}(f, x)$ constructed on the point-system*

$$\left\{\cos \frac{2k-1}{2n+1}\pi\right\}_{k=1}^{n+1}, \qquad \left\{\cos \frac{2k}{2n+1}\pi\right\}_{k=0}^{n}, \quad \text{or} \quad \left\{\cos \frac{k-1}{n-1}\pi\right\}_{k=1}^{n}$$

converges uniformly to every $f(x) \in C[-1, 1]$.

To prove our theorem we shall require the following

LEMMA. *For every $f(x) \in C[-1, 1]$, we have*

$$\lim_{n\to\infty} \frac{1}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1+x_k)} = \frac{f(-1)}{12}, \tag{1.9}$$

where

$$x_k = \cos \frac{2k-1}{2n+1}\pi, \qquad k = 1, 2,\ldots, n;$$

$$\lim_{n\to\infty} \frac{1}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1-x_k)} = \frac{f(1)}{12}, \tag{1.10}$$

where

$$x_k = \cos \frac{2k}{2n+1} \pi, \qquad k = 1, 2, \ldots, n;$$

and

$$\lim_{n\to\infty} \frac{1}{(n-1)^2} \sum_{k=1}^{n} \frac{x_k}{(1-x_k^2)} f(x_k) = \frac{f(1) - f(-1)}{6}, \tag{1.11}$$

where

$$x_k = \cos \frac{k-1}{n-1} \pi, \qquad k = 1, 2, \ldots, n.$$

Equalities (1.9) and (1.10) have been proved by the first author [2], while (1.11) has been established by Saxena [5].

2. Proof of the Theorem

Let

$$x_k = \cos \frac{2k-1}{2n+1} \pi, \qquad k = 1, 2, \ldots, n+1. \tag{2.1}$$

The points

$$\left\{\cos \frac{2k-1}{2n+1} \pi\right\}_{k=1}^{n}$$

are the zeros of the Jacobi polynomial $P_n^{(-1/2,1/2)}(x)$, which is identical with $[\cos(2n+1)\,\theta/2]/(\cos \theta/2)$, where $x = \cos\theta$, and which satisfies

$$(1-x^2)\, P_n''^{(-1/2,1/2)}(x) + (1-2x)\, P_n'^{(-1/2,1/2)}(x) + n(n+1)\, P_n^{(-1/2,1/2)}(x) = 0. \tag{2.2}$$

Let $W_n(x) = (1+x)\, P_n^{(-1/2,1/2)}(x)$. One easily sees that

$$W_n'(-1) = P_n^{(-1/2,1/2)}(-1), \qquad W_n''(-1) = 2P_n'^{(-1/2,1/2)}(-1)$$

$$W_n'(x_k) = (1+x_k)\, P_n'^{(-1/2,1/2)}(x_k), \tag{2.3}$$

$$W_n''(x_k) = (1+x_k)\, P_n''^{(-1/2,1/2)}(x_k) + 2P_n'^{(-1/2,1/2)}(x_k).$$

From (1.7), (2.2), and (2.3), we have

$$H^*_{\nu(n)}(f, x) = H_n(f, x) + \frac{(1+x)^2\,[P_n^{(-1/2,1/2)}(x)]^2}{(x - x_\nu)} \times \left[\frac{2}{(2n+1)^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1+x_k)} - \frac{2n(n+1)}{3(2n+1)} f(-1)\right], \quad (2.4)$$

which, on using Theorem 1 of [2], yields that, uniformly, $\lim_{n\to\alpha} H_n(f, x) = f(x)$ for $f(x) \in C[-1, 1]$, and (1.9) proves our theorem for the points

$$\left\{\cos \frac{2k-1}{2n+1}\,\pi\right\}_{k=1}^{n+1}.$$

For the other point-systems the proof follows similarly; we omit details.

Remark. Our theorem differs from that of Turán [6]. In our case the convergence is uniform in the whole interval $[-1, 1]$ for every $f(x) \in C[-1, 1]$ without any necessary and sufficient condition. In another paper we shall omit derivatives at more than one point.

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